Int J Theor Phys (2009) 48: 2214–2221
DOI 10.1007/s10773-009-0001-6

Bianchi-I Cosmologies with Electromagnetic and Spinor Fields. Isotropization Problem

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Received: 23 January 2009 / Accepted: 24 March 2009 / Published online: 15 April 2009

Abstract We consider Bianchi type I cosmologies with unidirectional magnetic and electric fields, assuming as well the existence of a global spinor field $\psi(t)$ as one more possible source of gravity able to suppress the inevitable anisotropy accompanying a nonzero vector field. The field $\psi(t)$ is assumed to contain a nonlinearity in the form s^n, where $s = \overline{\psi}\psi$ and $n = \text{const}$ (the special case $n = 1$ corresponds to a Dirac massive field). The structure of the stress-energy tensor of the spinor field is shown to be the same as that of a perfect fluid with the equation of state $p = w\rho$ where $w = n - 1$. The Dirac massive spinor field and nonlinear fields with $n < 4/3$ are shown to be able to provide isotropization. A numerical estimate shows that this isotropization could occur early enough to be compatible with observations.

Keywords Bianchi-I cosmology · Electromagnetic field · Spinor field · Isotropization

1 Introduction

The existence of intergalactic magnetic fields is known from observations. Their influence on various astrophysical phenomena and on the cosmological evolution has been studied for over four decades from both theoretical and observational points of view (see reviews by Giovannini [4], Wainwright [6], Zeldovich et al. [10] and references therein, and Harrison [5] for one of the first discussions of the role of magnetic fields in the early Universe). Cosmologists admit that such a field could be primordial in origin, i.e., could have appeared before nucleosynthesis and even before inflation.

A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction. The presently observed

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Universe is almost isotropic at large, therefore the isotropization problem appears inevitably in any study of anisotropic cosmologies. The simplest of such models, which nevertheless rather completely describe the anisotropy effects, are Bianchi type I homogeneous models whose spatial sections are flat but the expansion or contraction rate is direction-dependent.

We consider Bianchi-I cosmologies with magnetic (and also possibly electric) fields in the presence of one more source of gravity, namely, a global spinor field $\psi(t)$ which may be a linear (Dirac) field or contain a nonlinearity of the form $\Phi(s) = \lambda s^n$, $S = \overline{\psi}\psi$, $\lambda = \text{const}$, $n = \text{const}$. Such cosmological spinor fields are by now discussed comparatively rarely, see, e.g., Saha [8] and references therein. One should note, however, that one of the first researchers who stressed an important role of spinor fields in cosmology was J.A. Wheeler (Brill and Wheeler [2], Wheeler [9]).

We will show that the space-time anisotropy, inevitable in the presence of the electric and magnetic fields, can be successfully suppressed by a linear spinor field as well as by nonlinear fields with the exponent $n < 4/3$. In our problem setting, for such nonlinear fields, the stress-energy tensor structure is the same as that of a perfect fluid with the pressure to density ratio $w = p/\rho = n - 1$, and, under the condition $n < 4/3$, the spinor energy density decreases more slowly than that of the electromagnetic field and dominates at late times. The limiting case $n = 0$ ($w = -1$) corresponds to a cosmological constant and to an asymptotically de Sitter expansion.

We conclude that the spinor field is more efficient in anisotropy suppression than a dilatonic scalar field, interacting with the electromagnetic field by the law $F(\phi)F^{\mu\nu}F_{\mu\nu}$. The latter has been shown to lead to isotropization in Bianchi-I models under some special (fine-tuning) conditions only (Bronnikov et al. [3]), while in some previous papers it was even asserted that isotropization in such models was impossible at all.

One should also mention the papers (Bali and Gokhroo [1]) which considered a cosmological model with a magnetic field and a perfect fluid, and (Pradhan and Prakash Pandey [7]) where, alongside with a magnetic field, a neutral fluid with bulk viscosity was introduced and the influence of viscosity on the late-time expansion process was studied. Though, general conclusions like ours were not formulated there.

2 Field Equations

The Lagrangian of the field system is taken on the form

$$\begin{aligned} L &= \frac{R}{2\varkappa} + L_{\text{em}} + L_{\text{sp}} \\ &= \frac{R}{2\varkappa} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\left(\overline{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\overline{\psi}\gamma^\mu\psi\right) - m\overline{\psi}\psi - \Phi(s), \end{aligned} \tag{1}$$

where R is the scalar curvature, $\varkappa = 8\pi G$ is Einstein's constant of gravity, γ^μ are the Dirac matrices in curved space-time (so that $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$), $\overline{\psi}$ is the Dirac conjugate spinor, $s = \overline{\psi}\psi$, $m = \text{const}$, and $\Phi(s)$ is an arbitrary function.

The Lagrangian (1) leads to the Einstein equations

$$R_\mu^\nu = -\varkappa\left(T_\mu^\nu - \frac{1}{2}\delta_\mu^\nu T_\alpha^\alpha\right), \tag{2}$$

the Maxwell equations

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \tag{3}$$

and the spinor field equation

$$i\gamma^{\mu}\nabla_{\mu}\psi - m\psi - \frac{d\Phi}{ds}\psi = 0. \tag{4}$$

In (2), T_{μ}^{ν} is the total stress-energy tensor (SET) consisting of the spinor ($\underset{\rm sp}{T}{}_{\mu}^{\nu}$) and electromagnetic ($\underset{\rm e}{T}{}_{\mu}^{\nu}$) parts:

$$T_{\mu}^{\nu} = \underset{\rm sp}{T}{}_{\mu}^{\nu} + \underset{\rm e}{T}{}_{\mu}^{\nu}, \tag{5}$$

$$\underset{\rm sp}{T}{}_{\mu}^{\nu} = \frac{i}{4} g^{\nu\rho}\Big(\overline{\psi}\gamma_{\mu}\nabla_{\rho}\psi + \overline{\psi}\gamma_{\rho}\nabla_{\mu}\psi - \nabla_{\mu}\overline{\psi}\gamma_{\rho}\psi - \nabla_{\rho}\overline{\psi}\gamma_{\mu}\psi\Big) - \delta_{\mu}^{\nu} L_{\rm sp}, \tag{6}$$

$$\underset{\rm e}{T}{}_{\mu}^{\nu} = -2F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{2}\delta_{\mu}^{\nu} F_{\alpha\beta}F^{\alpha\beta}. \tag{7}$$

We consider the field system (1) in a Bianchi type I space-time, with the coordinates $(x^{\mu}) = (t,\ x,\ y,\ z)$ and the metric

$$\begin{aligned} ds^2 &= \mathrm{e}^{2\alpha}dt^2 - \mathrm{e}^{2\beta_1}dx^2 - \mathrm{e}^{2\beta_2}dy^2 - \mathrm{e}^{2\beta_3}dz^2 \\ &\equiv \mathrm{e}^{2\alpha}dt^2 - a_1^2dx^2 - a_2^2dy^2 - a_3^2dz^2, \end{aligned} \tag{8}$$

where α and β_i ($i = 1, 2, 3$) are functions of the time coordinate $x^0 = t$. The proper physical time of a comoving observer, $t = \tau$, corresponds to the coordinate condition $\alpha \equiv 0$. Another convenient time variable, the harmonic time coordinate $t = u$, is obtained if the lapse function $\mathrm{e}^{2\alpha}$ is chosen so that

$$\alpha = \beta_1 + \beta_2 + \beta_3. \tag{9}$$

This choice significantly simplifies the field equations. In terms of u, the nonzero components of the Ricci tensor are

$$R_0^0 = \mathrm{e}^{-2\alpha}\left(\ddot{\alpha} - \dot{\alpha}^2 + \sum_{i=1}^{3}\dot{\beta_i}^{2}\right), \qquad R_{\underline{i}}^{i} = \mathrm{e}^{-2\alpha}\ddot{\beta}_i. \tag{10}$$

There is no summing over an underlined index, and the dot denotes d/du.

We assume that there are homogeneous electric and magnetic fields having the same direction x. So the electromagnetic vector potential is taken in the form

$$A_{\mu} = \{0,\ A_1(t),\ 0, A_3(y)\}.$$

Then (3) and the corresponding Bianchi identities lead to

$$F_{01} = -F_{10} = \frac{q_e}{\sqrt{g}}\mathrm{e}^{2\alpha+2\beta_1}, \qquad F_{23} = -F_{32} = q_m, \tag{11}$$

where $g = |\det(g_{\mu\nu})|$, q_e and q_m are constants characterizing the electric and magnetic field intensities, respectively; other $F_{\mu\nu}$ are zero. Accordingly, the tensor $\underset{\rm e}{T}{}_{\mu}^{\nu}$ has the form

$$\underset{\rm e}{T}{}_{0}^{0} = \underset{\rm e}{T}{}_{1}^{1} = -\underset{\rm e}{T}{}_{2}^{2} = -\underset{\rm e}{T}{}_{3}^{3} = B^2 + E^2, \tag{12}$$

where E and B are the electric and magnetic field strengths:

$$E^2 = F_{01}F^{10} = q_e^2 \mathrm{e}^{-2\beta_2-2\beta_3}, \qquad B^2 = F_{23}F^{23} = q_m^2 \mathrm{e}^{-2\beta_2-2\beta_3}. \tag{13}$$

Thus the electromagnetic SET can be written as

$$\underset{\mathrm{e}}{T}{}_\mu^\nu = \frac{Q^2}{2a_2^2a_3^2}\mathrm{diag}(1,\ 1,\ -1,\ -1), \quad Q^2 = q_e^2 + q_m^2. \tag{14}$$

The spinor field equation (4) in the metric (8) is written as

$$i\overline{\gamma}^0\left(\partial_t + \frac{1}{2}\frac{\dot{v}}{v}\right)\psi - \mathrm{e}^\alpha \Psi_s \psi = 0, \tag{15}$$

where $v \equiv a_1a_2a_3 \equiv \mathrm{e}^{\beta_1+\beta_2+\beta_3}$ is the volume factor of the Universe, and the dot denotes d/dt; furthermore, $\Psi \equiv m\psi + \Phi$ and $\Psi_s \equiv d\Psi/ds$. In components, $\psi(t) = \{V_a(t)\}$, $a = 1, 2, 3, 4$, (15) has the form

$$\begin{aligned} &\dot{V}_r + \frac{1}{2}\frac{\dot{v}}{v}V_r + i\mathrm{e}^\alpha \Psi_s V_r = 0, \quad r = 1, 2; \\ &\dot{V}_l + \frac{1}{2}\frac{\dot{v}}{v}V_l - i\mathrm{e}^\alpha \Psi_s V_l = 0, \quad l = 3, 4, \end{aligned} \tag{16}$$

which may be integrated to give

$$\begin{aligned} V_r(t) &= \frac{C_r}{\sqrt{v}}\exp\left(-\int \Psi_s \mathrm{e}^\alpha dt\right), \\ V_l(t) &= \frac{C_l}{\sqrt{v}}\exp\left(\int \Psi_s \mathrm{e}^\alpha dt\right), \end{aligned} \tag{17}$$

where C_r and C_l are integration constants.

From (16) we get an equation for $s = V_1^*V_1 + V_2^*V_2 - V_3^*V_3 - V_4^*V_4$ and its solution:

$$\dot{s} + \frac{\dot{v}}{v}s = 0, \qquad s = \frac{s_0}{v(t)}, \qquad s_0 = \mathrm{const}, \qquad v = a_1a_2a_3 \equiv \mathrm{e}^{\beta_1+\beta_2+\beta_3}. \tag{18}$$

Then, using (15), we find

$$L_{\mathrm{sp}} = s\Psi_s - \Psi(s) = s\Phi_s - \Phi(s), \tag{19}$$

whence it follows for the spinor SET:

$$\underset{\mathrm{sp}}{T}{}_0^0 = \Psi(s) \equiv ms + \Phi(s), \qquad \underset{\mathrm{sp}}{T}{}_i^i = -L_{\mathrm{sp}} = \Phi(s) - \Phi_s. \tag{20}$$

Note that (11)–(20) are written in a form independent of the choice of the temporal coordinate in (8), and in what follows they can be used with any time gauge.

Let us now advert to the Einstein equations for the metric (8), using the harmonic gauge (9). The harmonic coordinate u and the proper physical time τ are connected in terms of the (so far unknown) volume factor $v(u) = v(\tau)$:

$$\tau = \int v(u)du, \quad \text{or} \quad u = \int \frac{d\tau}{v(\tau)}. \tag{21}$$

Since, for our system, $T^2_2 = T^3_3$, the difference $\binom{2}{2} - \binom{3}{3}$ of the Einstein equations (2) gives

$$\ddot\beta_2 - \ddot\beta_3 = 0 \quad \Rightarrow \quad \beta_2 - \beta_3 = C_1 u + D_1, \tag{22}$$

where C_1 and D_1 are integration constants, and we can put $D_1 = 0$ by properly choosing the zero point of the coordinate u, so that

$$\beta_2 - \beta_3 = C_1 u, \quad \text{or} \quad a_2/a_3 = \mathrm{e}^{C_1 u}. \tag{23}$$

As other two independent Einstein equations, one can choose the sum of spatial components $\binom{i}{i}$ of (2) and the temporal component $R^0_0 - \frac{1}{2}R = -\varkappa T^0_0$ (an analogue of the Friedmann equation), which may be written as

$$\mathrm{e}^{-2\alpha}\ddot\alpha = \varkappa\left[\frac{Q^2}{2a_2^2a_3^2} + 3(s\Psi_s - \Psi)\right], \tag{24}$$

$$\mathrm{e}^{-2\alpha}\sum_{i<k}\dot\beta_i\dot\beta_k = \varkappa\left(\frac{Q^2}{2a_2^2a_3^2} + \Psi\right), \tag{25}$$

or, in terms of the physical time τ,

$$\frac{v''}{v} = \varkappa\left[\frac{Q^2}{2a_2^2a_3^2} + 3(s\Psi_s - \Psi)\right], \tag{26}$$

$$H_1H_2 + H_1H_3 + H_2H_3 = \varkappa\left(\frac{Q^2}{2a_2^2a_3^2} + \Psi\right), \tag{27}$$

where the prime denotes $d/d\tau$ while $H_i(\tau) \equiv a_i'/a_i$ are the directional Hubble parameters of the model.

It is hard to obtain exact solutions to (26) and (27), even for particular choices of $\Phi(s)$. Therefore, in what follows we will only discuss the asymptotic behaviour of the model and the conditions under which its expansion becomes isotropic at late times when, by assumption, $v \to \infty$.

An exact solution for pure electromagnetic field as a source of gravity is easily obtained in terms of the harmonic coordinate u and may be written, e.g., as a special case ($\phi \equiv 0,\ \lambda = 0$) of the solution (13), (32) of (Bronnikov [3]), namely,

$$\beta_2 = -\beta_1 + c_2 u, \qquad \beta_3 = -\beta_1 + c_3 u, \qquad \alpha = -\beta_1 + (c_2 + c_3)u, \tag{28}$$

$$\mathrm{e}^{\beta_1} = \frac{k}{Q\cosh(ku + ku_0)}, \tag{29}$$

where $c_2,\ c_3,\ u_0 = \text{const}$ and two more integration constants have been removed by constant rescalings of the y- and z-axes. The essential integration constants $k > 0$, c_2 and c_3 are related by

$$k^2 = c_2c_3, \tag{30}$$

so that c_2 and c_3 have the same sign. Let them be positive, then, as $u \to \infty$, the volume factor v behaves as

$$v = \mathrm{e}^{\alpha} \sim \mathrm{e}^{hu}, \quad h = c_2 + c_3 + \sqrt{c_2c_3}, \tag{31}$$

and evidently the proper time is $\tau \sim v \to \infty$. One sees that at late times the model is extremely anisotropic: while $a_1 = e^{\beta_1} \to 0$, the other two scale factors a_2 and a_3 infinitely grow as well as the volume factor v. This stresses the necessity of other kinds of matter to make the model approach isotropy at large τ.

3 Isotropization Conditions

Isotropization means, in essence, that at large physical times τ, when the volume factor $v = v(\tau) = a_1 a_2 a_3 = e^{\beta_1+\beta_2+\beta_3}$ tends to infinity, the three scale factors $a_i(\tau) = e^{\beta_i}$ grow at the same rate. We will therefore say, by definition, that a model is isotropizing if

$$a_i/a \to \text{const} > 0 \quad \text{as } \tau \to \infty, \tag{32}$$

where $a(\tau) = v^{1/3}$ is the mean scale factor. Then, by rescaling some of the coordinates, we can make $a_i/a \to 1$, and the metric will become manifestly isotropic at large τ. The condition (32) implies for the directional Hubble parameters H_i

$$H_i - H_k \to 0 \quad \text{as } \tau \to \infty. \tag{33}$$

Other isotropization criteria and their relation to (32) are discussed in Appendix A of Bronnikov [3]. It can be verified that, for all asymptotic solutions considered here and satisfying (32), all other isotropization criteria hold as well.

Now, to obtain necessary conditions for isotropization, let us assume that it does occur as $\tau \to \infty$ and $v \to \infty$. This means

$$a_1 \sim a_2 \sim a_3 \sim v^{1/3} \to \infty, \qquad s = s_0/v \to 0 \quad \text{as } \tau \to \infty. \tag{34}$$

Let us take the asymptotic form of $\Phi(s)$ at small s in the form

$$\Phi(s) = \lambda s^n = \frac{\lambda s_0^n}{v^n}, \qquad n = \text{const}, \tag{35}$$

where $\lambda = \text{const}$ plays the role of a nonlinearity parameter. The further consideration depends on the parameters m, n and λ.

1. A linear (Dirac) spinor: $\lambda = 0$, $m > 0$. Then, as follows from (20), the spinor SET has the form $\mathrm{diag}(ms,\ 0,\ 0,\ 0)$, coinciding with that of dustlike matter with the density $\rho = ms$. Accordingly, with (34), the conservation equation $\nabla_\nu \underset{\text{sp}}{T}{}^\nu_0 = 0$ leads to $\rho \sim a^{-3}$ whereas the components of the manifestly anisotropic electromagnetic SET (14) decay faster with growing $a(\tau)$: $\underset{\text{e}}{T}{}^\nu_\mu \sim a^{-4}$. Thus the spinor SET dominates at late times and can provide isotropization. The asymptotic form of the solution to the Einstein equations is the same as in the spatially flat Friedmann model with dustlike matter, $a \sim \tau^{2/3}$.
2. If, in addition to the term ms, there is a nonlinearity with $n > 1$, the linear term still dominates at small s in the function $\Psi = ms + \Phi$, leading to the same conclusions as in item 1.
3. If there is a nonlinearity with $n < 1$, then $\Phi \gg ms$ at small s, and the spinor SET behaves as that of a perfect fluid with the equation of state $p = w\rho$, $w = n - 1$, and $\rho \sim a^{-3n}$. Again the spinor SET dominates over $\underset{\text{e}}{T}{}^\nu_\mu \sim a^{-4}$, and isotropization is possible. The Einstein equations lead to $a(\tau) \sim \tau^{2/(3n)}$. In case $n < 2/3$, this corresponds to power-law inflation. The limiting case $n = 0$ evidently corresponds to a cosmological constant, $\Lambda = \varkappa\Phi = \text{const}$, and the late-time asymptotic is de Sitter, $a(t) \sim \exp\left(\sqrt{\Lambda/3}\,\tau\right)$.

4. A nonlinearity with $n > 1$ can play an essential role at small s if there is no linear term in $\Psi(\psi)$, i.e., $m = 0$. In this case, the spinor SET again behaves as that of a perfect fluid with $w = n - 1$, and this "fluid" density is $\rho \sim a^{-3n}$. The spinor SET dominates over $\underset{\mathrm{e}}{T}{}_{\mu}^{\nu} \sim a^{-4}$, hence isotropization is possible if and only if $n < 4/3$. The asymptotic behavior of $a(\tau)$ is again $a \sim \tau^{2/(3n)}$ and is now non-inflationary.

The above isotropization conditions are only necessary but not sufficient. Thus, in principle, isotropization could be prevented according to (23) if $C_1 \neq 0$ and $u \to \infty$ as $\tau \to \infty$. However, it is not the case in the presently considered models: indeed, by (21), $u \sim \int d\tau/v$ while in all cases of interest $v \sim \tau^{2/n}$ with $n < 4/3$, so that $2/n > 3/2$, and the variable u tends to a finite limit at large τ. Thus we do not obtain any more restrictions other than those described in items 1–4.

4 A Numerical Estimate

Let us make a rough numerical estimate showing that the models under consideration, despite their oversimplified nature, can in principle be viable, which in our case should mean that sufficient isotropy was already achieved by the cosmological recombination time, at redshifts $z \sim 1000$.

Let us take, for simplicity, the model with a linear Dirac field and identify it with the present baryonic component of the matter content of the Universe, and a uniform magnetic field strength corresponding to the present-day intergalactic magnetic field strength $B \sim 10^{-6}$ Gs (Giovannini [4]), and let us find the value of the scale factor a at which their energy densities coincided. This time instant could conditionally be regarded as an estimate of the beginning of the isotropic expansion of the Universe. We thus ignore the dark energy, which could hardly affect the events at such an early stage of the evolution, and dark matter, which, having a dustlike equation of state, could provide an even earlier isotropization that we may find. So our estimate will be very conservative.

The baryonic matter density is well estimated as $\rho_{\mathrm{sp}} = \rho_m \approx 3 \times 10^{-31}$ g/cm^3 $\approx 8 \times 10^6$ cm^{-4} in natural units in which $c = \hbar = 1$. On the other hand, the magnetic field energy density ρ_{magn}, corresponding to $B = 10^{-6}$ Gs, is approximately 2.5×10^{-14} erg/cm^3 $\approx$ 800 cm^{-4} in the same units. Thus the present-day ratio of densities is

$$\rho_{\mathrm{sp}}/\rho_{\mathrm{magn}} \approx 10^4. \tag{36}$$

The spinor field energy density decreases as a^{-3} as the Universe expands, whereas that of the magnetic field as a^{-4}. Therefore the ratio (36) grows proportionally to $a(\tau)$ and was equal to unity when a/a_0 was about 10^{-4}, where a_0 is the present value of a, in other words, at $z \approx 10^4$.

Thus even if we extrapolate the intergalactic magnetic field density to the whole Universe (which is certainly a strong exaggeration), assume that this field is unidirectional and take into account only the baryonic (spinor) matter (all these assumptions worsen the estimate), we still obtain an early enough isotropization.

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